

Quintessence, unified dark energy and dark matter, and confinement/deconfinement mechanism

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ABSTRACT

We describe a new type of generalized gravity-matter models where $f(R) = R + R^2$ gravity couples in a non-conventional way to a scalar “inflaton” field, to a second scalar “darkon” field responsible for dark energy/dark matter unification, as well as to a non-standard nonlinear gauge field system containing a square-root of the ordinary Maxwell Lagrangian, which is responsible for a charge confining/deconfining mechanism. The essential non-conventional feature of our models is employing the formalism of non-Riemannian volume forms, *i.e.* metric-independent non-Riemannian volume elements on the spacetime manifold, defined in terms of auxiliary antisymmetric tensor gauge fields. Although being (almost) pure-gauge degrees of freedom, the non-Riemannian volume-forms trigger a series of important features unavailable in ordinary gravity-matter models. Upon passing to the physical Einstein frame we obtain an effective matter-gauge-field Lagrangian of quadratic “k-essence” type both w.r.t. the “inflaton” and the “darkon”, with the following properties: (i) Remarkable effective “inflaton” potential possessing two infinitely large flat regions with vastly different heights (“vacuum” energy densities) describing the “early” and “late” Universe; (ii) Nontrivial effective gauge coupling constants running with the “inflaton”, in particular, effective “inflaton”-running coupling constant of the square-root Maxwell term, which determines the strength of the charge confinement; (iii) The confinement-strength gauge coupling constant is non-zero in the “late” Universe, *i.e.*, charge confinement is operating, whereas it vanishes in the “early” Universe, *i.e.*, confinement-free epoch; (iv) The unification of dark energy and dark matter is explicitly seen within the FLRW reduction, where they appear as dynamically generated effective vacuum energy density and dynamically induced dust-like matter, correspondingly.

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1. Introduction

Our main task, which pertains to the interface of particle physics and cosmology [1], is to present a non-canonical model of extended ($f(R) = R + R^2$) [2] gravity interacting in a non-standard way with two scalar fields and a strongly nonlinear gauge field, which is capable to provide a systematic description of:

(a) Unified treatment of dark energy and dark matter (for a background, see [3, 4]) revealing them as manifestations of a single material entity – the first non-canonical scalar “darkon” field. A number of proposals already exist for an adequate description of dark energy’s and dark matter’s dynamics within the framework of standard general relativity or its modern extensions, among them: “Chaplygin gas” models [5], “purely kinetic k-essence” models [6], “mimetic” dark matter models [7]. In Section 2 we briefly review our own approach [8].

(b) Quintessential scenario driven by the remarkable dynamically generated effective potential of the second “inflaton” scalar field with a consistent explanation of the vast difference between the energy scales of the “early” and the “late” Universe;

(c) Charge confinement/deconfinement mechanism triggered via special interplay between the “inflaton” and the nonlinear gauge field dynamics explaining absence of charge confinement in the “early” Universe and exhibiting confinement in the “late” Universe.

The principal ingredient of our approach is the method of non-Riemannian volume-forms (metric-independent volume elements) on the pertinent spacetime manifold (see refs.[9, 10, 11] for a consistent geometrical formulation, which is an extension of the originally proposed method [12]). Non-Riemannian volume-forms are constructed in terms of auxiliary maximal-rank antisymmetric tensor gauge fields, which are shown to be essentially pure-gauge degrees of freedom, *i.e.*, they do not introduce additional propagating field-theoretic degrees of freedom. Yet, they leave a trace in a form of several dynamically induced integration constants, which trigger a series of important features unavailable in ordinary gravity-matter models.

Another important ingredient of our approach is the inclusion of interaction of the extended gravity with a nonlinear gauge field system containing alongside the ordinary Maxwell Lagrangian also a square-root of the latter. In flat spacetime such gauge field system is known [13, 14] to yield a simple implementation of ‘t Hooft’s idea [15] about confinement being produced due to the presence in the energy density of electrostatic field configurations of a term *linear* w.r.t. electric displacement field in the infrared region (presumably as an appropriate infrared counterterm). It has been shown in [13] (for flat spacetime) and in appendix B of [11] (in curved spacetime) that the strength of confinement in this model is measured by the corresponding coupling constant of the square-root Maxwell term. Let us also note that one could start as well with the non-Abelian version of the above nonlinear gauge theory containing square-root of the Yang-Mills

Lagrangian. For static spherically symmetric solutions, the non-Abelian theory effectively reduces to an Abelian one as pointed out in [13].

In Section 3 we obtain upon passing to the physical Einstein frame an effective matter-gauge-field Lagrangian of quadratic “k-essence” type [16] w.r.t. both the “inflaton” and the “darkon” scalar fields with several remarkable properties:

(i) It contains an effective “inflaton” potential possessing two infinitely large flat regions with vastly different heights – dynamically generated “vacuum” energy densities thanks to the appearance of the above mentioned integration constants – which describe the “early” and “late” Universe, accordingly;

(ii) The Einstein-frame Lagrangian contains nontrivial effective gauge coupling constants running with the “inflaton”. Particularly important is the effective “inflaton”-running coupling constant of the square-root Maxwell term, which determines the “inflaton”-dependent strength of the charge confinement;

(iii) We show the confinement-strength gauge coupling constant to be non-zero in the “late” Universe, *i.e.*, charge confinement is operating there. On the other hand, the confinement-strength gauge coupling constant vanishes in the “early” Universe, *i.e.*, the latter being confinement-free epoch;

In Section 4 we consider a cosmological FLRW reduction of the Einstein-frame effective matter-gauge-field Lagrangian where the unification of dark energy and dark matter is explicitly seen upon identifying them as dynamically generated effective vacuum energy density and dynamically induced dust-like matter, correspondingly.

2. A Simple Model of Dark Energy and Dark Matter Unification

We start with a simple particular case of a non-conventional gravity-scalar-field action – a member of the general class of the non-Riemannian-volume-element-based gravity-matter theories [17, 11] (we use for simplicity units where the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x \sqrt{-g} R + \int d^4x (\sqrt{-g} + \Phi(C)) L(u, Y) . \tag{1}$$

R denotes the standard Riemannian scalar curvature for the spacetime Riemannian metric $g_{\mu\nu}$. $L(u, Y)$ is general-coordinate invariant Lagrangian of a single scalar field $u(x)$, the simplest example being:

$$L(u, Y) = Y - V(u) \quad , \quad Y \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu u \partial_\nu u , \tag{2}$$

It is coupled symmetrically to two mutually independent spacetime volume-elements (integration measure densities) – the standard Riemannian $\sqrt{-g}$

and to an alternative non-Riemannian (metric-independent) one:

$$\Phi(C) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu C_{\nu\kappa\lambda} . \quad (3)$$

As a result of the equations of motion w.r.t. $C_{\mu\nu\lambda}$ we obtain a crucial new property – *dynamical constraint* on $L(u, Y)$:

$$\partial_\mu L(u, Y) = 0 \longrightarrow L(u, Y) = -2M_0 = \text{const} , \text{ i.e. } Y = V(u) - 2M_0 . \quad (4)$$

The integration constant M_0 will play below (Eq.(5)) the role of dynamically generated cosmological constant.

The pertinent energy-momentum tensor $T_{\mu\nu}$ can be cast into a relativistic hydrodynamical form (taking into account (4)):

$$T_{\mu\nu} = -2M_0 g_{\mu\nu} + \rho_0 u_\mu u_\nu , \quad (5)$$

$$\rho_0 \equiv \left(1 + \frac{\Phi(C)}{\sqrt{-g}}\right) 2Y \quad , \quad u_\mu \equiv -\frac{\partial_\mu u}{\sqrt{2Y}} \quad , \quad u^\mu u_\mu = -1 , \quad (6)$$

with the corresponding pressure p and energy density ρ :

$$p = -2M_0 = \text{const} \quad , \quad \rho = \rho_0 - p = 2M_0 + \left(1 + \frac{\Phi(C)}{\sqrt{-g}}\right) 2Y . \quad (7)$$

Because of the constant pressure ($p = -2M_0$) the covariant energy-momentum conservation $\nabla^\nu T_{\mu\nu} = 0$ implies *both* conservation of a hidden Noether symmetry current $J^\mu = \rho_0 u^\mu$, as well as *geodesic fluid motion*:

$$\nabla_\mu J^\mu \equiv \nabla_\mu (\rho_0 u^\mu) = 0 \quad , \quad u_\nu \nabla^\nu u_\mu = 0 . \quad (8)$$

The *hidden strongly nonlinear Noether symmetry* giving rise to the current $J^\mu = \rho_0 u^\mu$ results from the invariance (up to a total derivative) of the scalar field action in (1), exclusively due to the presence of the non-Riemannian volume element $\Phi(C)$, under the following nonlinear symmetry transformations:

$$\delta_\epsilon u = \epsilon \sqrt{Y} \quad , \quad \delta_\epsilon g_{\mu\nu} = 0 \quad , \quad \delta_\epsilon C^\mu = -\epsilon \frac{1}{2\sqrt{Y}} g^{\mu\nu} \partial_\nu u (\Phi(C) + \sqrt{-g}) , \quad (9)$$

where $C^\mu \equiv \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} C_{\nu\kappa\lambda}$.

Thus, $T_{\mu\nu}$ (5)-(7) represents an exact sum of two contributions of the two “dark” material species with corresponding pressures and energy densities:

$$p = p_{\text{DE}} + p_{\text{DM}} \quad , \quad \rho = \rho_{\text{DE}} + \rho_{\text{DM}} \quad (10)$$

$$p_{\text{DE}} = -2M_0 \quad , \quad \rho_{\text{DE}} = 2M_0 \quad ; \quad p_{\text{DM}} = 0 \quad , \quad \rho_{\text{DM}} = \rho_0 , \quad (11)$$

i.e., according to (8) the dark matter component is a dust fluid flowing along geodesics. This is explicit unification of dark energy and dark matter originating from the non-canonical dynamics of a single scalar field – the “darkon” $u(x)$ coupled symmetrically to a standard Riemannian and another non-Riemannian (metric-independent) volume element. Let us also note that the physical result $T_{\mu\nu}$ (5) does *not* depend on the explicit form of the “darkon” potential $V(u)$.

Upon reduction to the cosmological class of Friedmann-Lemaitre-Robertson-Walker (FLRW) metrics:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (12)$$

the J^μ -current conservation (8) now reads:

$$\nabla^\mu(\rho_0 u_\mu) = 0 \quad \rightarrow \quad \frac{d}{dt}(a^3 \rho_0) = 0 \quad \rightarrow \quad \rho_0 = \frac{c_0}{a^3}, \quad c_0 = \text{const}, \quad (13)$$

$$p = -2M_0, \quad \rho = 2M_0 + \frac{c_0}{a^3}, \quad (14)$$

with the dark matter contribution ρ_0 being the typical cosmological dust solution (e.g. [18]).

3. Noncanonical Gravity-Matter System Coupled to Charge-Confining Nonlinear Gauge Field

Let us now extend the simple gravity-“darkon” model (1) to $f(R) = R + R^2$ gravity coupled non-canonically to both “inflaton” $\varphi(x)$ and “darkon” $u(x)$ scalar fields within the non-Riemannian volume-form formalism, as well as we will also add coupling to a non-standard non-linear gauge field subsystem:

$$S = \int d^4x \Phi(A) \left[g^{\mu\nu} R_{\mu\nu}(\Gamma) + X - V_1(\varphi) - \frac{1}{2} f_0 \sqrt{-F^2} \right] + \int d^4x \Phi(B) \left[\epsilon R^2 - \frac{1}{4e^2} F^2 + \frac{\Phi(H)}{\sqrt{-g}} \right] + \int d^4x (\sqrt{-g} + \Phi(C)) L(u, Y). \quad (15)$$

Here the following notations are used:

- $\Phi(A)$ and $\Phi(B)$ are two new metric-independent non-Riemannian volume-elements:

$$\Phi(A) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu A_{\nu\kappa\lambda}, \quad \Phi(B) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu B_{\nu\kappa\lambda}, \quad (16)$$

defined in terms of field-strengths of two auxiliary 3-index antisymmetric tensor gauge fields, apart from $\Phi(C)$;

- $\Phi(H) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu H_{\nu\kappa\lambda}$ is the dual field-strength of an additional auxiliary tensor gauge field $H_{\nu\kappa\lambda}$ whose presence is crucial for the consistency of (15).
- Let us specifically emphasize the importance to use here the Palatini formalism: $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$; $g_{\mu\nu}$, $\Gamma_{\mu\nu}^\lambda$ – metric and affine connection are *a priori* independent.
- The “inflaton” part reads:

$$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \quad , \quad V_1(\varphi) = f_1 \exp\{-\alpha\varphi\} . \quad (17)$$

- $F_{\mu\nu}$ is the field-strength of an (Abelian) gauge field \mathcal{A}_μ :

$$F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu \quad , \quad F^2 = F_{\mu\nu}F_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda} \quad (18)$$

- The coupling parameters f_1 , f_0 and α are dimensionful positive constants (mass^4 , mass^2 , mass^{-1}).

The specific form of the action (15) apart from the “darkon” field part (1) has been fixed by the requirement for invariance under global Weyl-scale transformations:

$$\begin{aligned} g_{\mu\nu} &\rightarrow \lambda g_{\mu\nu} \quad , \quad \Gamma_{\nu\lambda}^\mu \rightarrow \Gamma_{\nu\lambda}^\mu \quad , \quad \varphi \rightarrow \varphi + \frac{1}{\alpha} \ln \lambda \quad , \quad \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu \quad , \\ A_{\mu\nu\kappa} &\rightarrow \lambda A_{\mu\nu\kappa} \quad , \quad B_{\mu\nu\kappa} \rightarrow \lambda^2 B_{\mu\nu\kappa} \quad , \quad H_{\mu\nu\kappa} \rightarrow H_{\mu\nu\kappa} . \end{aligned} \quad (19)$$

As shown in [11], the solution to the equations of motion w.r.t. independent affine connection $\Gamma_{\nu\lambda}^\mu$ yield the latter as a Levi-Civita connection:

$$\Gamma_{\nu\lambda}^\mu = \Gamma_{\nu\lambda}^\mu(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa} (\partial_\nu\bar{g}_{\lambda\kappa} + \partial_\lambda\bar{g}_{\nu\kappa} - \partial_\kappa\bar{g}_{\nu\lambda}) \quad , \quad (20)$$

w.r.t. the Weyl-rescaled metric:

$$\bar{g}_{\mu\nu} = (\chi_1 + 2\varepsilon\chi_2 R)g_{\mu\nu} \quad , \quad \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}} \quad , \quad \chi_2 \equiv \frac{\Phi_2(B)}{\sqrt{-g}} . \quad (21)$$

Varying the action (15) w.r.t. auxiliary tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda}$ yields the equations:

$$\begin{aligned} \partial_\mu \left[R + X - V_1(\varphi) - \frac{1}{2}f_0\sqrt{-F^2} \right] &= 0 \\ \partial_\mu \left[\varepsilon R^2 - \frac{1}{4e^2}F^2 + \frac{\Phi(H)}{\sqrt{-g}} \right] &= 0 \quad , \quad \partial_\mu \left(\frac{\Phi_2(B)}{\sqrt{-g}} \right) = 0 \quad , \end{aligned} \quad (22)$$

whose solutions are:

$$\begin{aligned}
 R + X - V_1(\varphi) - \frac{1}{2}f_0\sqrt{-F^2} &= M_1 = \text{const} , \\
 \epsilon R^2 - \frac{1}{4e^2}F^2 + \frac{\Phi(H)}{\sqrt{-g}} &= -M_2 = \text{const} , \quad \frac{\Phi_2(B)}{\sqrt{-g}} \equiv \chi_2 = \text{const} . \quad (23)
 \end{aligned}$$

Here M_1 and M_2 are arbitrary dimensionful (mass⁴) and χ_2 arbitrary dimensionless integration constants. The appearance of M_1, M_2 signifies *dynamical spontaneous breakdown* of global Weyl-scale invariance under (19) due to the scale non-invariant solutions (second and third ones) in (23).

The physical meaning of the integration constants $M_0, M_{1,2}, \chi_2$, has been elucidated in refs.[10, 19, 11] from the point of view of the canonical Hamiltonian formalism. Namely, it was shown that $M_0, M_{1,2}, \chi_2$, which remain the only traces of the auxiliary gauge fields $C_{\mu\nu\lambda}, A_{\mu\nu\lambda}, B_{\mu\nu\lambda}, H_{\mu\nu\lambda}$, are identified as conserved Dirac-constrained canonical momenta conjugated to (certain components of) the latter.

Using the equations of motion of the starting action (15) w.r.t. $g_{\mu\nu}$ as well as Eqs.(23) we obtain the following relation between the original metric $g_{\mu\nu}$ and the Weyl-rescaled one (21) (with $\chi_{1,2}$ the same as in (21), Y - same as in (2)):

$$\bar{g}_{\mu\nu} = \chi_1 \Omega g_{\mu\nu} \quad , \quad \chi_1 \Omega = \frac{\chi_1 + 2\epsilon\chi_2(V_1(\varphi) + M_1)}{1 + 2\epsilon\chi_2(\bar{X} - \frac{f_0}{2}\sqrt{-\bar{F}^2})} , \quad (24)$$

$$\chi_1 = (V_1(\varphi) + M_1)^{-1} \left[2\chi_2 M_2 - 4M_0 - (1 + \chi_4)Y \right] , \quad (25)$$

where:

$$\bar{X} \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \quad , \quad \bar{F}^2 \equiv F_{\mu\nu}F_{\kappa\lambda}\bar{g}^{\mu\kappa}\bar{g}^{\nu\lambda} \quad , \quad \chi_4 \equiv \frac{\Phi(C)}{\sqrt{-g}} \quad (26)$$

Now, following the same steps as in refs.[11] we derive from (15) the physical *Einstein-frame* theory w.r.t. Weyl-rescaled Einstein-frame metric $\bar{g}_{\mu\nu}$ (24) and perform an additional “darkon” field redefinition $u \rightarrow \tilde{u}$:

$$\frac{\partial\tilde{u}}{\partial u} = (V_1(u) - 2M_0)^{-\frac{1}{2}} \quad ; \quad Y \rightarrow \tilde{Y} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\tilde{u}\partial_\nu\tilde{u} . \quad (27)$$

The explicit form of the Einstein-frame matter action reads:

$$\begin{aligned}
 L_{\text{eff}} &= \bar{X} - \frac{f_0}{2}\sqrt{-\bar{F}^2} - \frac{\chi_2}{e^2}\bar{F}^2 + \epsilon\chi_2\left(\bar{X} - \frac{f_0}{2}\sqrt{-\bar{F}^2}\right)^2 \\
 &\quad - \tilde{Y}\left(V(\varphi) + M_1\right)\left[1 + 2\epsilon\chi_2\left(\bar{X} - \frac{f_0}{2}\sqrt{-\bar{F}^2}\right)\right] \\
 &\quad + \tilde{Y}^2\left\{\chi_2\left[M_2 + \epsilon(V(\varphi) + M_1)^2\right] - 2M_0\right\} . \quad (28)
 \end{aligned}$$

To reveal the total matter effective potential U_{total} , i.e., the *minus* part of L_{eff} (28) for static (spacetime independent) matter fields φ , F^2 and u (not \tilde{u} (27)), we note that relations (4), (24) and (27) imply $\tilde{Y} = \frac{1}{\chi_1 \Omega}$, meaning that according to (24)-(25) for static matter fields:

$$\tilde{Y} \Big|_{\text{static}} = \frac{(V(\varphi) + M_1)(1 - \epsilon \chi_2 f_0 \sqrt{-\bar{F}^2})}{2 \left\{ \chi_2 \left[M_2 + \epsilon(V(\varphi) + M_1)^2 \right] - 2M_0 \right\}}. \quad (29)$$

Upon inserting (29) in (28) we find:

$$U_{\text{total}}(\varphi, \bar{F}^2) = -L_{\text{eff}} \Big|_{\text{static}} = \mathcal{U}(\varphi) + \frac{1}{2} f_{\text{eff}}(\varphi) \sqrt{-\bar{F}^2} + \frac{1}{4e_{\text{eff}}^2(\varphi)} \bar{F}^2, \quad (30)$$

$$\mathcal{U}(\varphi) \equiv \frac{(V_1(\varphi) + M_1)^2}{4 \left\{ \chi_2 \left[M_2 + \epsilon(V_1(\varphi) + M_1)^2 \right] - 2M_0 \right\}}, \quad (31)$$

(recall (17) $V_1(\varphi) = f_1 \exp\{-\alpha\varphi\}$),

with running “inflaton”-dependent gauge coupling constants:

$$f_{\text{eff}}(\varphi) = f_0(1 - 4\epsilon\chi_2\mathcal{U}(\varphi)), \quad \frac{1}{e_{\text{eff}}^2(\varphi)} = \frac{\chi_2}{e^2} \left[1 + \epsilon e^2 f_0^2 (1 - 4\epsilon\chi_2\mathcal{U}(\varphi)) \right] \quad (32)$$

Concluding this Section let us note that $f_{\text{eff}}(\varphi)$ measures the strength of charge confinement. Indeed, as shown in Appendix B of [11], for static spherically symmetric fields in a static spherically symmetric metric the presence of the term $-\frac{1}{2}f_{\text{eff}}(\varphi)\sqrt{-\bar{F}^2}$ will produce an effective “Cornell”-type [20] potential $V_{\text{eff}}(L)$ between charged quantized fermions, L being the distance between the latter:

$$V_{\text{eff}}(L) = -\frac{e_{\text{eff}}^2(\varphi)}{2\pi} \frac{1}{L} + e_{\text{eff}}(\varphi) f_{\text{eff}}(\varphi) \sqrt{2} L + (L\text{-independent const}), \quad (33)$$

i.e., a linear confining plus a Coulomb piece. Thus, we will consider below the dynamics of the Abelian nonlinear gauge system as mimicking quantum chromodynamics.

4. Cosmological Implications

4.1. Flat Regions of the Total “Inflaton” Effective Potential

The total matter effective potential (30) possesses non-trivial “inflaton”-dependent gauge field vacuum ($\frac{\partial}{\partial \bar{F}^2} U_{\text{total}} \Big|_{\bar{F}_{\text{vac}}^2} = 0$):

$$\sqrt{-\bar{F}_{\text{vac}}^2} = e_{\text{eff}}^2(\varphi) f_{\text{eff}}(\varphi), \quad (34)$$

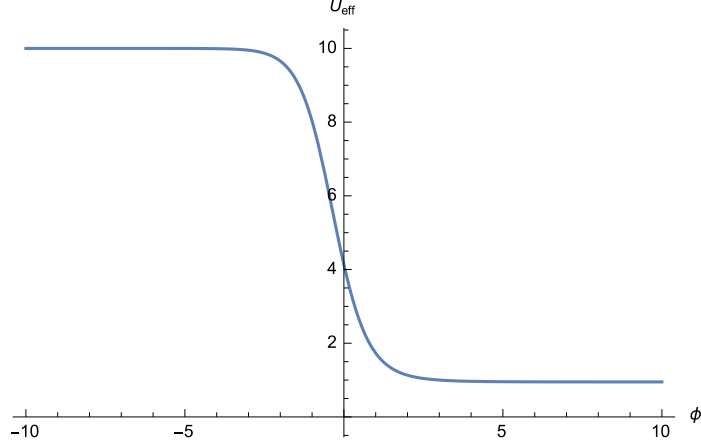


Figure 1: Qualitative shape of the total effective “inflaton” potential $U_{\text{eff}}(\varphi)$ (35).

Upon inserting (34) in (30) we get the following total effective “inflaton” potential $U_{\text{eff}}(\varphi)$ (taking into account expressions (31)-(32)):

$$U_{\text{eff}}(\varphi) = \mathcal{U}(\varphi) + \frac{1}{4}e_{\text{eff}}^2(\varphi)f_{\text{eff}}^2(\varphi) = \mathcal{U}(\varphi) + \frac{e^2 f_0^2 (1 - 4\epsilon\chi_2 \mathcal{U}(\varphi))^2}{4\chi_2 [1 + e^2 \epsilon f_0^2 (1 - 4\epsilon\chi_2 \mathcal{U}(\varphi))]} \quad (35)$$

As clearly seen from the graphic representation of (35) on Fig.1, the total effective “inflaton” potential possesses the following remarkable property – two infinitely large flat regions corresponding to large negative and large positive “inflaton” values (the sub-/super-scripts (\pm) label belonging to the respective flat region):

- $(-)$ flat region – for large negative φ -values:

$$\mathcal{U}(\varphi) \simeq \mathcal{U}_{(-)} = \frac{1}{4\epsilon\chi_2} \quad , \quad U_{\text{eff}}(\varphi) \simeq U_{\text{eff}}^{(-)} = \mathcal{U}_{(-)} = \frac{1}{4\epsilon\chi_2} \quad , \quad (36)$$

$$f_{\text{eff}}(\varphi) \simeq f_0^{(-)} = 0 \quad , \quad e_{\text{eff}}^2(\varphi) \simeq e_{(-)}^2 = \frac{e^2}{\chi_2} \quad . \quad (37)$$

The first relation in (37) implies according to (33) that there is **no charge confinement in the $(-)$ flat region.**

- $(+)$ flat region – for large positive φ -values:

$$\mathcal{U}(\varphi) \simeq \mathcal{U}_{(+)} = \frac{M_1^2}{4 [\chi_2 (M_2 + \epsilon M_1^2) - 2M_0]} \quad , \quad (38)$$

$$U_{\text{eff}}(\varphi) \simeq U_{\text{eff}}^{(+)} = \frac{\epsilon\chi_2 M_1^2 + \epsilon e^2 f_0^2 (\chi_2 M_2 - 2M_0)}{4\epsilon\chi_2 [(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) + \epsilon\chi_2 M_1^2]}, \quad (39)$$

$$f_{\text{eff}}(\varphi) \simeq f_0^{(+)} = f_0 \frac{(\chi_2 M_2 - 2M_0)}{\chi_2 M_2 - 2M_0 + \epsilon\chi_2 M_1^2}, \quad (40)$$

$$\frac{1}{e_{\text{eff}}^2(\varphi)} \simeq \frac{1}{e_{(+)}^2} = \frac{\chi_2}{e^2} \left[1 + \frac{\epsilon e^2 f_0^2 (\chi_2 M_2 - 2M_0)}{\chi_2 M_2 - 2M_0 + \epsilon\chi_2 M_1^2} \right]. \quad (41)$$

Relation (40) implies according to (33) that **charge confinement does take place in the (+) flat region**. Also, there is a non-trivial rescaling of the fundamental electric charge unit when passing from the (-) flat “inflaton” region (second relation in (37)) to the (+) flat “inflaton” region (41).

Let us specifically point out that both the vacuum energy densities $U_{\text{eff}}^{(\pm)}$ as well as the confinement/deconfinement phenomena in both flat “inflaton” regions are entirely *dynamically* induced.

Associating appropriately the integration constants $M_{1,2}$, M_0 and the coupling constants f_0 , ϵ with the fundamental physical constants M_{EW} (electroweak scale), M_{Pl} (Planck mass) and M_{infl} (inflationary energy scale):

$$M_1 \sim M_{EW}^4, \quad M_2 \sim M_{Pl}^4, \quad e^2 f_0^2 \sim \frac{M_1}{M_2}, \quad \chi_2 \epsilon \sim M_{infl}^{-4}, \quad (42)$$

allow us to identify – through their respective vacuum energy densities $U_{\text{eff}}^{(-)} \sim 10^{-8} M_{Pl}^4 \sim M_{infl}^4$ (39) and $U_{\text{eff}}^{(+)} \sim 10^{-122} M_{Pl}^4$ (36) – the (-) flat “inflaton” region as describing the “early” Universe, whereas the (+) flat “inflaton” region will be describing the “late” Universe in accordance with the PLANCK collaboration data [21].

4.2. FLRW Reduction and Dark Energy/Dark Matter Unification

Let us now consider a reduction of the Einstein-frame gauge-matter action (28) to the class of cosmological FLRW class of metrics (12) (here for simplicity we will take $K = 0$ – flat spacial sections):

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}, \quad (43)$$

where now $\bar{X} = \frac{1}{2} \dot{\varphi}^2(t)$, $\bar{Y} = \frac{1}{2} \dot{u}^2(t)$, and $\sqrt{-F^2} = \dot{A}(t)$. The resulting full FLRW Einstein-frame action reads (in the gauge $N(t) = 1$):

$$S_{\text{FLRW}} = \int dt \left[-6a \dot{a}^2 + a^3 L_{\text{FLRW}} \right] \quad (44)$$

with the FLRW matter-gauge Lagrangian, or the pressure p :

$$\begin{aligned}
 p \equiv L_{\text{FLRW}} = & -\frac{f_0}{2} \left[1 + \epsilon\chi_2 \left(\dot{\varphi}^2 - \dot{\tilde{u}}^2 (V_1(\varphi) + M_1) \right) \right] \dot{A} \\
 & + \frac{\chi_2}{4e^2} (1 + \epsilon e^2 f_0^2) \dot{A}^2 + \frac{\dot{\varphi}^2}{2} + \epsilon\chi_2 \frac{\dot{\varphi}^4}{4} - \frac{\dot{\tilde{u}}^2}{2} (V_1(\varphi) + M_1) (1 + \epsilon\chi_2 \dot{\varphi}^2) \\
 & + \frac{\dot{\tilde{u}}^4}{4} \left[\chi_2 M_2 - 2M_0 + \epsilon\chi_2 (V_1(\varphi) + M_1)^2 \right]. \quad (45)
 \end{aligned}$$

The canonical FLRW Hamiltonian corresponding to $L_{\text{FLRW}} \equiv p$ (45), or the energy density ρ is given by:

$$\rho = \frac{\pi_A}{a^3} + \frac{p_\varphi}{a^3} + \frac{p_{\tilde{u}}}{a^3} - L_{\text{FLRW}}, \quad (46)$$

where the canonical momenta are defined via:

$$\frac{\pi_A}{a^3} = \frac{\partial L_{\text{FLRW}}}{\partial \dot{A}}, \quad \frac{p_\varphi}{a^3} = \frac{\partial L_{\text{FLRW}}}{\partial \dot{\varphi}}, \quad \frac{p_{\tilde{u}}}{a^3} = \frac{\partial L_{\text{FLRW}}}{\partial \dot{\tilde{u}}}. \quad (47)$$

From the first Eq.(47) we find:

$$\dot{A} = \frac{2e^2}{\chi_2(1 + \epsilon e^2 f_0^2)} \left\{ \frac{\pi_A}{a^3} + \frac{f_0}{2} \left[1 + \epsilon\chi_2 \left(\dot{\varphi}^2 - \dot{\tilde{u}}^2 (V_1(\varphi) + M_1) \right) \right] \right\}, \quad (48)$$

whereas the second and the third Eqs.(47) are a system of two mixed cubic algebraic equations w.r.t. the velocities $\dot{\varphi}$ and $\dot{\tilde{u}}$:

$$\begin{aligned}
 \frac{p_\varphi}{a^3} = & \frac{\epsilon\chi_2 \dot{\varphi}^3}{1 + \epsilon e^2 f_0^2} + \frac{\dot{\varphi}}{1 + \epsilon e^2 f_0^2} \left[1 - 2\epsilon e^2 f_0 \frac{\pi_A}{a^3} - \epsilon\chi_2 (V_1(\varphi) + M_1) \dot{\tilde{u}}^2 \right], \quad (49) \\
 \frac{p_{\tilde{u}}}{a^3} = & \dot{\tilde{u}}^3 \left[\chi_2 M_2 - 2M_0 + \frac{\epsilon\chi_2}{1 + \epsilon e^2 f_0^2} (V_1(\varphi) + M_1)^2 \right] \\
 & - \dot{\tilde{u}} \frac{(V_1(\varphi) + M_1)}{1 + \epsilon e^2 f_0^2} \left(1 - 2\epsilon e^2 f_0 \frac{\pi_A}{a^3} - \epsilon\chi_2 \dot{\varphi}^2 \right). \quad (50)
 \end{aligned}$$

(i) In the (+) flat region of the total effective “inflaton” potential, i.e. in the “late” Universe where φ is large positive, neglecting $V_1(\varphi) = f_1 e^{-\alpha\varphi}$ in (49)-(50), we obtain approximate (up to higher powers of $\frac{1}{a^3}$) solutions for the velocities $\dot{\varphi}$ and $\dot{\tilde{u}}$ as functions of the canonical momenta:

$$\dot{\varphi} = \frac{p_\varphi}{a^3} (1 + \epsilon e^2 f_0^2) \frac{(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) + \epsilon\chi_2 M_1^2}{(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2)} + \mathcal{O}(a^{-6}), \quad (51)$$

$$\begin{aligned} \dot{\tilde{u}} = & \left[\frac{M_1}{(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) + \epsilon \chi_2 M_1^2} \right]^{\frac{1}{2}} + \frac{1}{2}(1 + \epsilon e^2 f_0^2) \times \\ & \times \left[M_1 \left((\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) + \epsilon \chi_2 M_1^2 \right) \right]^{-\frac{1}{2}} \frac{p_{\tilde{u}}}{a^3} + \mathcal{O}(a^{-6}). \end{aligned} \quad (52)$$

Inserting (51)-(52) into (45)-(46) we obtain for the pressure and energy density in the “late” Universe:

$$p_{(+)} = -U_{\text{eff}}^{(+)} + \mathcal{O}(a^{-6}), \quad (53)$$

$$\begin{aligned} \rho_{(+)} = & U_{\text{eff}}^{(+)} + \left\{ p_{\tilde{u}} \left[\frac{M_1}{(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) + \epsilon \chi_2 M_1^2} \right]^{\frac{1}{2}} \right. \\ & \left. + \pi_A \frac{e^2 f_0}{\chi_2(1 + \epsilon e^2 f_0^2)} \frac{(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) - \epsilon \chi_2 M_1^2}{(\chi_2 M_2 - 2M_0)(1 + \epsilon e^2 f_0^2) + \epsilon \chi_2 M_1^2} \right\} \frac{1}{a^3} + \mathcal{O}(a^{-6}), \end{aligned} \quad (54)$$

where $U_{\text{eff}}^{(+)}$ is the dynamically induced vacuum energy density in the “late” Universe (39). Note that there is *no* $\mathcal{O}(a^{-3})$ in the expression for the pressure (53), while the leading terms in (53)-(54) are with opposite signs. Thus, comparing with (14) we can identify the expressions (53)-(54) as describing a unification in the “late” Universe of the dark energy ($\rho_{(+)}^{\text{DE}} = -p_{(+)}^{\text{DE}} = U_{\text{eff}}^{(+)}$) and dust dark matter ($p_{(+)}^{\text{DM}} = 0$, $\rho_{(+)}^{\text{DM}} = \mathcal{O}(a^{-3})$) term in (54)) contributions..

(ii) In the (–) flat region of the total effective “inflaton” potential, i.e. in the “early” Universe where φ is large negative, the approximate solutions for $\dot{\varphi}$ and $\dot{\tilde{u}}$ as functions of the canonical momenta – counterparts of Eqs.(51)-(52) – read:

$$\dot{\varphi} = \left[\frac{(1 + \epsilon e^2 f_0^2) p_{\varphi}}{2\epsilon \chi_2} \right]^{1/3} a^{-1} + \mathcal{O}(a^{-2}), \quad (55)$$

$$\dot{\tilde{u}} = \frac{1}{\sqrt{\epsilon \chi_2 f_1}} - \frac{1}{2} \sqrt{\frac{\epsilon \chi_2}{f_1}} \left[\frac{(1 + \epsilon e^2 f_0^2) p_{\varphi}}{2\epsilon \chi_2} \right]^{2/3} a^{-2} + \mathcal{O}(a^{-3}). \quad (56)$$

Inserting (55)-(56) into (45)-(46) we obtain for the pressure and energy density in the “early” Universe:

$$p_{(-)} = -U_{\text{eff}}^{(-)} + \mathcal{O}(a^{-4}), \quad (57)$$

$$\rho_{(+)} = U_{\text{eff}}^{(-)} + \frac{p_{\tilde{u}} e^{\frac{1}{2}\varphi}}{\sqrt{\epsilon \chi_2 f_1}} a^{-3} + \mathcal{O}(a^{-4}), \quad (58)$$

where $U_{\text{eff}}^{(-)}$ is the dynamically induced vacuum energy density in the “early” Universe (36). Once again we see that there is *no* $\mathcal{O}(a^{-3})$ in the expression

for the pressure (57) as in (53), while the leading terms in (57)-(58) are with opposite signs. Thus, comparing with (14) we can again identify the expressions (57)-(58) as describing a unification in the “early” Universe of the dark energy ($\rho_{(-)}^{\text{DE}} = -p_{(-)}^{\text{DE}} = U_{\text{eff}}^{(-)}$) and dust dark matter ($p_{(-)}^{\text{DM}} = 0$, $\rho_{(-)}^{\text{DM}} = O(a^{-3})$ term in (58)) contributions.

5. Conclusions

We have demonstrated above that the method of non-Riemannian spacetime volume-forms (metric-independent volume elements), combined with other ingredients such as non-canonical gravity interactions with “inflaton” and “darkon” scalar fields and special type of non-linear gauge fields, provides a plausible description of various basic features of cosmological evolution:

(a) Unified description of dark energy and dark matter – through the impact of the dynamics of the non-canonical scalar “darkon” field;

(b) Natural explanation of the huge difference between the vacuum energy scales in the “early” and “late” Universe – thanks to the generation of dimensionful integration constants as an impact of the dynamics of the non-Riemannian volume-form fields;;

(c) Natural explanation of the absence of charge confinement in the “early” Universe epoch, and the appearance of QCD-like confinement in the “late” Universe - through an “inflaton”-running coupling constants in the physical Einstein-frame theory.

Further physically relevant phenomena adequately described using the method of non-Riemannian spacetime volume-forms are:

(d) A novel mechanism for the supersymmetric Brout-Englert-Higgs effect, namely, the appearance of a dynamically generated cosmological constant triggering spontaneous supersymmetry breaking and mass generation for the gravitino [9, 19].

(e) Gravity-assisted spontaneous symmetry breaking of electroweak gauge symmetry – when adding interactions with the bosonic fields of the electroweak sector of the standard model of elementary particles gravity triggers the generation of a Higgs-like spontaneous symmetry breaking effective potential in the “late” Universe, whereas in the “early” Universe gravity keeps the Higgs-like scalar isodoublet massless, *i.e.*, no spontaneous electroweak breaking in the “early” Universe [22].

Finally, let us mention some other modifications of the method of non-Riemannian volume elements – the so called gravity models with dynamical spacetime [23], which were further developed into models of interacting diffusive unified dark energy and dark matter ([24] and references therein).

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